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LETTER TO THE EDITOR

Level crossings and commuting observables for the quantum elliptic billiard

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Abstract. The energy spectrum of the quantum elliptic billiard is obtained by solving the Schrödinger equation in elliptic coordinates. The corresponding eigenstates also diagonalise an operator B which commutes with H. A numerical search of the exact eigenvalues of B and H permits one to follow each state as a function of the deformation parameter μ. Geometrical arguments, also valid for the simpler problem of the rectangular box, allow us to understand the results obtained.

The analysis of energy spectra of different physical bound systems has always been an interesting subject in quantum mechanics. The energy-level spacings as well as the existence of degeneracies (crossings) in such systems have been widely studied during recent years. The number of degrees of freedom, the separability of the problem and the number of free parameters involved, among others, are very important features which have to be taken into account when a given quantum spectrum is analysed.

Starting from Percival's ideas (Percival 1973) for systems with more than one degree of freedom, two kinds of spectra have been distinguished. The regular spectrum, whose level spacing is characterised by a Poisson distribution (Berry and Tabor 1977), is associated with integrable problems (the harmonic oscillator is an exception). The other one, the irregular spectrum characterised by a Gaussian distribution, corresponds to non-integrable systems (Pechukas 1983, Yukawa 1985).

This statistical behaviour can be related to crossings and repulsions between levels with the same symmetry when the spectra are analysed as functions of the parameters of the problem. In fact, integrable systems depending on one parameter exhibit many crossings while non-integrable systems show repulsions and double-hyperbola curves, rather than degeneracies.

In a recent paper, Arvieu and Ayant (1987) studied a physical example of such integrable systems. It is the problem of a spinless particle in a hard elliptic box (quantum elliptical billiard). The spectrum is presented as a function of the deformation parameter $\mu = a/b$ (a and b being the major and minor axes of the ellipse) while the area is preserved constant. It exhibits a lot of crossings but also a certain number of repulsions (see for instance figure 9 of Arvieu and Ayant) which drew the attention of the authors and which in fact do not really exist. The authors also stated in their

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paper that the real existence of the crossings would have to be proved numerically and that an analysis similar to that done by Berry (Berry and Wilkinson 1984) is still lacking.

The aim of this letter is to display the correct behaviour of the energy levels and to explain the spectrum using simple geometrical considerations. In fact it has been known for some time that levels of this kind of system will typically cross when μ is varied. However, the diagonalisation procedure adopted by Arvieu and Ayant does not allow one to follow each level continuously and leads to incorrect identifications of the states. In the present work we will solve the Schrödinger equation in elliptic coordinates, developing an alternative method for computing the exact eigenvalues. Moreover, we will simultaneously diagonalise a new observable B which commutes with H, providing two eigenvalues which characterise each state.

Let us first consider the Schrödinger equation in elliptic coordinates. It is easy to observe that it can be separated in two Mathieu equations (Stratton 1941)

$$\sqrt{\xi^2-1}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sqrt{\xi^2-1}\,\frac{\mathrm{d}}{\mathrm{d}\xi}f_1\right)+(\sigma^2k^2\xi^2-b)f_1=0 \tag{1a}$$

$$\sqrt{1-\eta^2}\,\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\sqrt{1-\eta^2}\,\frac{\mathrm{d}}{\mathrm{d}\eta}f_2\right)+(b-\sigma^2k^2\eta^2)f_2=0 \tag{1b}$$

where $k^2=2mE/\hbar^2$, $\sigma=a(1-\mu^2)^{1/2}$ is the focal distance and b is the constant of separation. In order to have physical solutions it is necessary that $f_2(\eta)$ be a periodic function of v ($\eta=\cos v$). The values of b satisfying this condition are the characteristic values of the Mathieu equation and constitute a denumerable set $b_0, b_1, b_2, \ldots$ for the even solutions and $b_1, b_2, b_3, \ldots$ for the odd solutions.

In order to understand the physical meaning of b we will turn on to the classical equations. The classical Hamiltonian in elliptic coordinates can be written

$$H=\frac{1}{2m\sigma^2(\xi^2-\eta^2)}[(\xi^2-1)p_\xi^2+(1-\eta^2)p_\eta^2]+V(\xi,\eta) \tag{2}$$

where $p_\xi=\partial\mathscr{L}/\partial\dot{\xi}$ and $p_\eta=\partial\mathscr{L}/\partial\dot{\eta}$ are the classical momenta and $\mathscr{L}$ is the classical Lagrangian. For $\xi<\xi_0$ (ξ_0 determines the boundary of the ellipse), $V(\xi,\eta)=0$. Thus, a second constant of motion can be found such that

$$-(\xi^2-1)p_\xi^2+2m\sigma^2E\xi^2-B=0 \tag{3a}$$

$$-(1-\eta^2)p^2-2m\sigma^2E\eta^2+B=0. \tag{3b}$$

It is easy to relate the constant B to the product of the angular momenta with respect to the focii at $x=\pm\sigma$

$$\boldsymbol{l}_1\cdot\boldsymbol{l}_2=\frac{(\xi^2-\eta^2)[(\xi^2-1)\dot{\eta}^2-(1-\xi^2)\dot{\xi}^2]m^2\sigma^4}{(\xi^2-1)(1-\eta^2)} \tag{4}$$

so that

$$B=\boldsymbol{l}_1\cdot\boldsymbol{l}_2+2m\sigma^2E. \tag{5}$$

If $\boldsymbol{l}_1\cdot\boldsymbol{l}_2>0$, $B>2m\sigma^2E$ and the classical trajectory repeatedly touches an ellipse characterised by $\xi_{\mathrm{lim}}=B/2m\sigma^2E$. On the other hand, if $\boldsymbol{l}_1\cdot\boldsymbol{l}_2<0$, the trajectory always lies between the focii touching a hyperbola determined by $\eta_{\mathrm{lim}}=B/2m\sigma^2E$ (see also Berry 1981). In this last case, from (3*b*) B is a positive number; thus we obtain a lower limit for $\boldsymbol{l}_1\cdot\boldsymbol{l}_2$ given by $-2m\sigma^2E$.

Turning to the quantum mechanical problem, the physically allowed values of the separation constant b, i.e. the characteristic values of the Mathieu equation, are the eigenvalues (in units of $\hbar^2$) of a new observable $\check{B}$ which commutes with $\check{H}$

$$\check{B} = \check{l}_1 \cdot \check{l}_2 + 2m\sigma^2 \check{H}. \tag{6}$$

The integer which enumerate these characteristic values indicates the number n_η of nodes of the wavefunction corresponding to the pseudo-angular variable η. The resulting eigenenergies are also labelled by another quantum number n_ξ, which accounts for the number of nodes of the wavefunction of the pseudo-radial variable ξ.

The wavefunction factorises in the form

$$\psi^a_{n_\xi, n_\eta}(\xi, \eta) = R^a_{n_\xi, n_\eta}(\xi) S^a_{n_\xi, n_\eta}(\eta) \tag{7}$$

where R is the first-kind radial Mathieu function and S is the angular Mathieu function. The parity of the Mathieu functions is indicated by whether the superscript a is an even or odd number (Stratton 1941).

For computing this wavefunction we used the algorithm 352 from the ACM library (Clemm 1969). However, numerical difficulties in the determination of the eigenvalues of $\check{B}$ and $\check{H}$ arise from the fact that the boundary condition for ξ (i.e. $R(\xi_0) = 0$), which determines the eigenenergies, is coupled with the boundary condition for η (i.e. the periodicity condition) which determines the allowed values of b for a given E.

To satisfy simultaneously both conditions a numerical self-consistent method was developed. Assuming that $R^{E,b(E)}(\xi_0) = 0$ we can expand R for $E \simeq E_0$ so that

$$R^{E,b(E)}(\xi_0) \simeq R'|_{E=E_0} \Delta E \tag{8}$$

where

$$R'|_{E=E_0} = \frac{R^{E+\delta, b(E+\delta)}(\xi_0) - R^{E,b(E)}(\xi_0)}{\delta} \tag{9}$$

and in consequence we can derive ΔE for an initial value of E. The second iteration starts with $E - \Delta E$ and repeating the procedure we obtain a quick convergence (i.e. $|\Delta E|/E \sim 10^{-6}$) finding E_0 and $b(E_0)$.

Figure 1 shows the eigenenergies obtained for the first states as a function of $\boldsymbol{l}_1 \cdot \boldsymbol{l}_2$ for a deformation parameter $\mu = 0.6$. From this figure we observe that in the classical limit the eigenvalues of $\boldsymbol{l}_1 \cdot \boldsymbol{l}_2$ have a minimum value of $-2m\sigma^2 E$. Another interesting feature arises from the fact that for high energies a lot of degeneracies appear between the even-x, even-y $(++)$ parity states and the odd-x, odd-y $(--)$ parity states. The same holds for the $(+-)$ and the $(-+)$ states.

Figure 2(a) displays the exact energy levels for the elliptic billiard (2a) for the even-x and even-y parity states as a function of μ. We have verified that a similar behaviour holds for the states with other parities. A very simple interpretation of these results can be done in terms of geometrical arguments. Keeping the area constant and starting from $\mu = 1$ (circular box) when μ increases the pseudoradial quanta are hardened (the wavelength associated with the motion of ξ decreases) and the pseudoangular quanta are softened (the corresponding wavelength increases). Figure 3 illustrates this fact. Thus we can understand the three types of levels appearing in the spectrum.

(i) Levels with $n_\eta = 0$ and $n_\xi \neq 0$ are monotonously increasing.

(ii) Levels with $n_\xi = 0$ and $n_\eta \neq 0$ are monotonously decreasing.

(iii) Mixed levels which initially could decrease but, starting from a given deformation, will always increase. For small values of μ the softening of the η quanta could

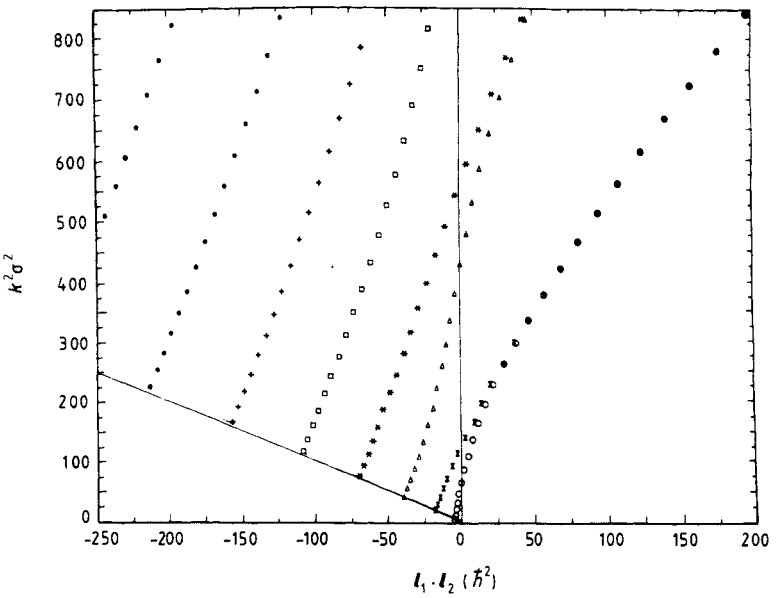

Figure 1. Energy eigenvalues as a function of $l_1 \cdot l_2$ corresponding to the eigenfunctions of the elliptic billiard which simultaneously diagonalise B and H. The open circles (○) correspond to the even solutions with $n_\xi = 0$ (see text) and the crosses (×) to the odd solutions with $n_\xi = 0$. The triangles (△) and asterisks (*) indicate the even and odd solutions, respectively, with $n_\xi = 1$ and the squares (□) and the plus signs (+) correspond to the eigenfunctions with $n_\xi = 2$. Finally, states with $n_\xi = 3$ are shown with full circles (●). For each of these sets of eigenfunctions, characterised by n_ξ and the parity, the quantum number n_η increases with energy.

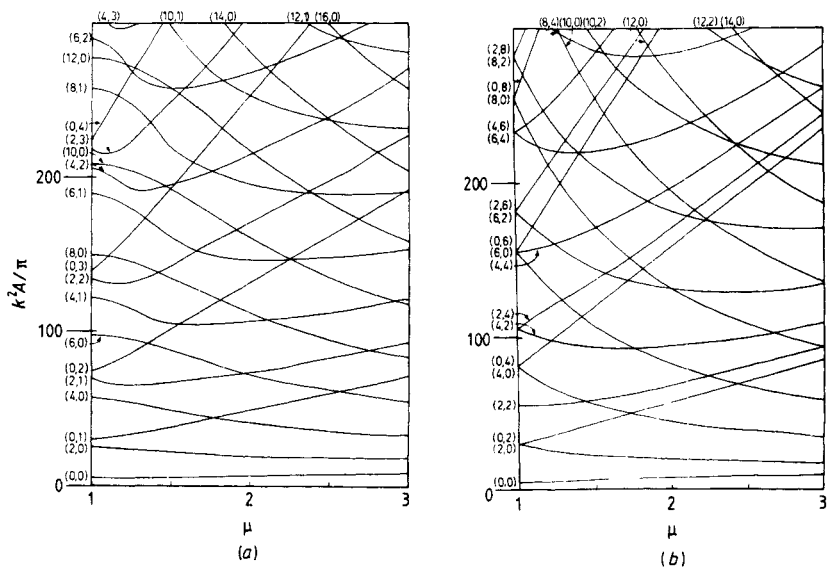

Figure 2. Energy spectra for (*a*) the elliptical and (*b*) the rectangular billiard corresponding to even-even parity states which are labelled by (n_η, n_ξ) and (n_x, n_y) respectively.

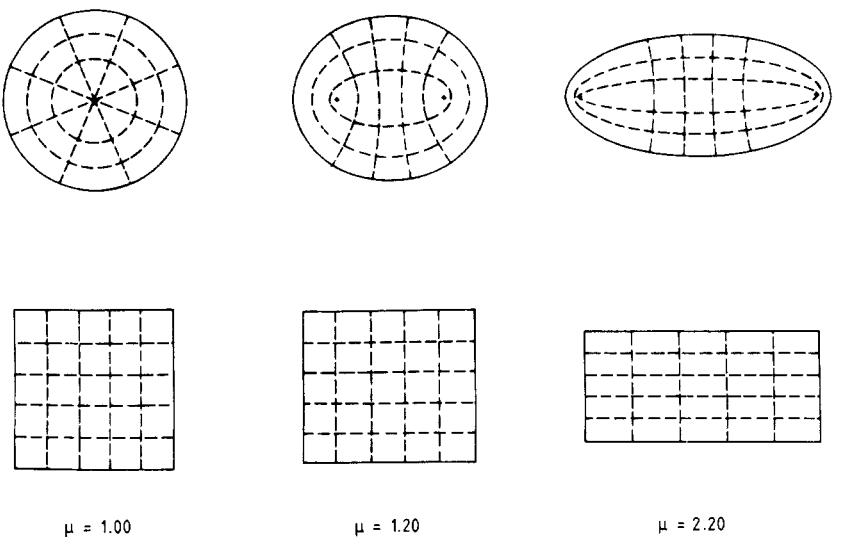

Figure 3. Shapes of a rectangular box and an elliptical box as a function of the deformation parameter μ ($\mu = D_x/D_y$ for the rectangle and $\mu = a/b$ for the ellipse). The area is preserved constant. For the elliptical case, the position of the nodes for the ($n_\eta = 4$, $n_\xi = 2$) eigenstate are drawn with broken lines. The corresponding nodes for the ($n_x = 4$, $n_y = 4$) eigenstate of the rectangle are also shown.

dominate but starting from a given μ (which depends on n_ξ and n_η) the hardening of the ξ quanta will finally prevail.

In this way, the crossings and degeneracies in the spectrum can be easily understood. Indeed, the double hyperbola behaviour will never be present because the energies increase (respectively, decrease) monotonously for the states (i) (respectively (ii)) or they decrease for small μ and then go to infinity for states (iii).

This qualitative understanding of the spectrum can also be applied to other two-dimensional integrable systems where a deformation parameter can be defined, like, for instance, the well known rectangular billiard. In this latter case the corresponding eigenenergies with respect to the ground state as a function of $\mu = D_x/D_y$ (D_x and D_y being the sides of the rectangle) follow the rule

$$E(n_x, n_y, \mu) \sim (1/\mu)[(n_x+1)^2-1] + \mu[(n_y+1)^2-1] \tag{10}$$

where n_x and n_y are the number of nodes of the wavefunction in the x and y directions. The ground state corresponds to $n_x = n_y = 0$. This spectrum is displayed in figure 2(b) together with those of the elliptic billiard. We can also appreciate the three different behaviours associated with the hardening of the n_y quanta and the softening of the n_x quanta when μ increases, which is also illustrated in figure 3. The similarity between figures 2(a) and 2(b) confirms the previous discussion.

Finally, it is interesting to relate the crossings of figure 2 with the degeneracies which can be present in two-degrees-of-freedom systems with two parameters μ_1 and μ_2 (Berry and Wilkinson 1984). When these systems remain integrable for all μ_1 and μ_2, the preceding arguments can be applied for each value of the parameter μ_2, when the energies are studied as a function of μ_1. Thus the surfaces $E(\mu_1, \mu_2)$ will cross in continuum curves leading to diedric intersections. On the other hand, if the system turns on a non-integrable problem, accidental degeneracies of the surface $E(\mu_1, \mu_2)$

could appear, leading to diabolic points. Only in such cases would it be necessary to perform a detailed analysis, as the one done by Berry and Wilkinson, to confirm the phase factor which acquires the wavefunction when the parameters μ_1 and μ_2 change adiabatically, enclosing the crossing point.

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